University of Wrocław: Data Science 15.09.2020

Entrance exam: Probability theory

First name: ..... Last name: .....

1. (**6 pts**) Random vector (X, Y) has a density

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (0,1)^2, \\ 0 & \text{otherwise.} \end{cases}$$

   Compute $E[\min(X, Y)]$.

2. (**7 pts**) Let X be a Poisson random variable with parameter $\lambda_1 = 1$ and Y be a Poisson random variable (independent from X) with parameter $\lambda_2 = 6$. (Recall, Z is a Poisson r.v. with parameter $\lambda > 0$ if $Pr(Z = k) = \frac{\lambda^k e^{-\lambda}}{k!}$, for $k = 0, 1, 2, \ldots$) Assume that we know that $X + Y = 70$. What is the expected value of X given that $X + Y = 70$? More precisely, compute ET, where

$$P(T = k) = P(X = k | X + Y = 70), \quad k = 0, \ldots, 70.$$

3. Assume we sample $n = 26$ independent random variables $X_1, \ldots, X_n$, where each $X_i \in \{\mathsf{A}, \mathsf{C}, \mathsf{G}, \mathsf{T}\}, i = 1, \ldots, n$ has a distribution:

$$P(X_i = \mathsf{A}) = \frac{1}{2}, \quad P(X_i = \mathsf{C}) = P(X_i = \mathsf{G}) = P(X_i = \mathsf{T}) = \frac{1}{6}.$$

   Let Y be a random variable denoting the number of occurences of the pattern AGA.

   a. (**3 pts**) What is the expectation of Y, i.e., EY ?

   b. (**4 pts**) What is the second moment of Y, i.e., $E(Y^2)$?

   Note that an overlapping is allowed, *e.g.*, a sequence (of length 5) AGAGA contains two (overlapping) occurences of the pattern.